

# Pomeron in the $\mathcal{N} = 4$ supersymmetric gauge model at strong couplings

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## Abstract

We find the BFKL Pomeron intercept at  $N = 4$  super-symmetric gauge theory in the form of the inverse coupling expansion  $j_0 = 2 - 2\lambda^{-1/2} - \lambda^{-1} + 1/4 \lambda^{-3/2} + 2(1 + 3\zeta_3)\lambda^{-2} + O(\lambda^{-5/2})$  with the use of the AdS/CFT correspondence in terms of string energies calculated recently. The corresponding slope  $\gamma'(2)$  of the anomalous dimension calculated directly up to the fifth order of perturbation theory turns out to be in an agreement with the closed expression obtained from the recent Basso results.

*PACS:* 12.38.Bx

# 1 Introduction

Pomeron is the Regge singularity of the  $t$ -channel partial wave [1] responsible for the approximate equality of total cross-sections for high energy particle-particle and particle-antiparticle interactions valid in an accordance with the Pomeronchuk theorem[2]. In QCD the Pomeron is a colorless object, constructed from reggeized gluons [3].

The investigation of the high energy behavior of scattering amplitudes in the  $\mathcal{N} = 4$  Supersymmetric Yang-Mills (SYM) model [4, 5, 6] is important for our understanding of the Regge processes in QCD. Indeed, this conformal model can be considered as a simplified version of QCD, in which the next-to-leading order (NLO) corrections [7] to the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [3] are comparatively simple and numerically small. In the  $N = 4$  SYM the equations for composite states of several reggeized gluons and for anomalous dimensions of quasi-partonic operators turn out to be integrable at the leading logarithmic approximation [8, 9]. Further, the eigenvalue of the BFKL kernel for this model has the remarkable property of the maximal transcendentality [5]. This property gave a possibility to calculate the anomalous dimensions (AD)  $\gamma$  of the twist-2 Wilson operators in one [10], two [5, 11], three [12], four [13, 14] and five [15] loops using the QCD results [16] and the asymptotic Bethe ansatz [17] improved with wrapping corrections [14] <sup>1</sup> in an agreement with the BFKL predictions [4, 5].

On the other hand, due to the AdS/CFT-correspondence [19, 20, 21], in  $\mathcal{N} = 4$  SYM some physical quantities can be also computed at large couplings. In particular, for AD of the large spin operators Beisert, Eden and Staudacher constructed the integral equation [22] with the use of the asymptotic Bethe-ansatz. This equation reproduced the known results at small coupling constants and is in a full agreement (see [23, 24]) with large coupling predictions [25, 26].

With the use of the BFKL equation in a diffusion approximation [3, 4, 6], strong coupling results for AD [25] and the pomeron-graviton duality [27] the Pomeron intercept was calculated at the leading order in the inverse coupling constant (see the Erratum[28] to the paper [12]). <sup>2</sup> Similar results in the  $\mathcal{N} = 4$  SYM and QCD were obtained in Refs. [29] and [30]. The Pomeron-graviton duality in the  $\mathcal{N} = 4$  SYM gives a possibility to construct the Pomeron interaction model as a generally covariant effective theory for the reggeized gravitons [31].

Below we use recent calculations [32, 33, 34, 35] of string energies to find the strong coupling corrections to the Pomeron intercept  $j_0 = 2 - \Delta$  in next orders. We discuss also the relation between the Pomeron intercept and the slope of the anomalous dimension at  $j = 2$ .

## 2 BFKL equation at small coupling constant

The eigenvalue of the BFKL equation in  $\mathcal{N} = 4$  SYM model has the following perturbative expansion [4, 5] (see also Ref. [6])

$$j - 1 = \omega = \frac{\lambda}{4\pi^2} \left[ \chi(\gamma_{BFKL}) + \delta(\gamma_{BFKL}) \frac{\lambda}{16\pi^2} \right], \quad \lambda = g^2 N_c, \quad (1)$$

where  $\lambda$  is the t'Hooft coupling constant. The quantities  $\chi$  and  $\delta$  are functions of the conformal weights  $m$  and  $\bar{m}$  of the principal series of unitary Möbius group representations, but for the

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<sup>1</sup>The anomalous dimensions up to four loops were calculated also with the use of the Baxter equation [18].

<sup>2</sup>The value of this intercept was estimated earlier in Ref.[11].

conformal spin  $n = m - \widetilde{m} = 0$  they depend only on the BFKL anomalous dimension

$$\gamma_{BFKL} = \frac{m + \widetilde{m}}{2} = \frac{1}{2} + i\nu \quad (2)$$

and are presented below [4, 5]

$$\chi(\gamma) = 2\Psi(1) - \Psi(\gamma) - \Psi(1 - \gamma), \quad (3)$$

$$\delta(\gamma) = \Psi''(\gamma) + \Psi''(1 - \gamma) + 6\zeta_3 - 2\zeta_2\chi(\gamma) - 2\Phi(\gamma) - 2\Psi(1 - \gamma). \quad (4)$$

Here  $\Psi(z)$  and  $\Psi'(z)$ ,  $\Psi''(z)$  are the Euler  $\Psi$ -function and its derivatives. The function  $\Phi(\gamma)$  is defined as follows

$$\Phi(\gamma) = 2 \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k + \gamma} \beta'(k + 1), \quad (5)$$

where

$$\beta'(z) = \frac{1}{4} \left[ \Psi'\left(\frac{z+1}{2}\right) - \Psi'\left(\frac{z}{2}\right) \right]. \quad (6)$$

Due to the symmetry of  $\omega$  to the substitution  $\gamma_{BFKL} \rightarrow 1 - \gamma_{BFKL}$  expression (1) is an even function of  $\nu$

$$\omega = \omega_0 + \sum_{m=1}^{\infty} (-1)^m D_m \nu^{2m}, \quad (7)$$

where

$$\omega_0 = 16 \ln 2 \frac{\lambda}{4\pi^2} \left[ 1 + c_1 \frac{\lambda}{16\pi^2} \right] + O(\lambda^3), \quad (8)$$

$$D_m = 8 \left( 2^{2m+1} - 1 \right) \zeta_{2m+1} \frac{\lambda}{4\pi^2} + \frac{\delta^{(2m)}(1/2)}{(2m)!} \frac{\lambda^2}{64\pi^4} + O(\lambda^3). \quad (9)$$

According to Ref. [5] we have

$$c_1 = -\frac{1}{2 \ln 2} \left( 11\zeta_3 - 32 \text{Ls}_3\left(\frac{\pi}{2}\right) - \frac{165}{16} \pi \zeta_2 \right) \approx -7.5912, \quad (10)$$

where (see [36])

$$\text{Ls}_3(x) = -\int_0^x \ln^2 \left| 2 \sin\left(\frac{y}{2}\right) \right| dy. \quad (11)$$

Thus, the rightmost Pomeron singularity of the partial wave  $f_j(t)$  in the perturbation theory is situated at

$$j_0 = 1 + \omega_0 = 1 + 16 \ln 2 \frac{\lambda}{4\pi^2} \left[ 1 + c_1 \frac{\lambda}{16\pi^2} \right] + O(\lambda^3) \quad (12)$$

for small values of coupling  $\lambda$ . In turn, the anomalous dimension  $\gamma$  also has the square root singularity in this point, which means, that the convergency radius of the perturbation series in  $\lambda$  for the anomalous dimension  $\gamma = \gamma(\omega, \lambda)$  at small  $\omega$  is given by the expression

$$\lambda_{cr} = \frac{\pi^2 \omega}{4 \ln 2} \left( 1 - c_1 \frac{\omega}{64 \ln 2} \right) + O(\omega^3).$$

It will be interesting to find higher order corrections to the BFKL intercept  $j_0$  from the investigation of convergency of the perturbation theory using the analytic results for the anomalous dimensions obtained recently [5, 12, 13, 14, 15]. Note, that the BFKL singularity for positive  $\omega$  is situated at positive  $\lambda = \lambda_{cr}$ . But it is expected, that with growing  $\omega$  the nearest singularity, responsible for the perturbation theory divergency will be at negative  $\lambda$ . Positions of both singularities can be found from the perturbative expansion of  $\gamma$  with the possible use of appropriate resummation methods (cf. [12]).

Due to the Möbius invariance and hermicity of the BFKL hamiltonian in  $N = 4$  SUSY expansion (7) is valid also at large coupling constants. In the framework of the AdS/CFT correspondence the BFKL Pomeron is equivalent to the reggeized graviton [27]. In particular, in the strong coupling regime  $\lambda \rightarrow \infty$

$$j_0 = 2 - \Delta, \quad (13)$$

where the leading contribution  $\Delta = 2/\sqrt{\lambda}$  was calculated in Refs. [28, 29, 30]. Below we find next-to-leading terms in the strong coupling expansion of the Pomeron intercept. In the next section the simple approach to the intercept estimates discussed shortly in Ref. [28] will be reviewed.

### 3 AdS/CFT correspondence

Due to the energy-momentum conservation, the universal anomalous dimension of the stress tensor  $T_{\mu\nu}$  should be zero, i.e.,

$$\gamma(j=2) = 0. \quad (14)$$

It is important, that the anomalous dimension  $\gamma$  contributing to the DGLAP equation [37] does not coincide with  $\gamma_{BFKL}$  appearing in the BFKL equation. They are related as follows [7] (see also [38])

$$\gamma = \gamma_{BFKL} + \frac{\omega}{2} = \frac{j}{2} + i\nu, \quad (15)$$

where the additional contribution  $\omega/2$  is responsible in particular for the cancelation of the singular terms  $\sim 1/\gamma^3$  obtained from the NLO corrections (1) to the eigenvalue of the BFKL kernel [7].

Using above relations one obtains

$$\nu(j=2) = i. \quad (16)$$

As a result, from eq. (7) for the Pomeron intercept we derive the following representation for the correction  $\Delta$  (13) to the graviton spin 2

$$\Delta = \sum_{m=1}^{\infty} D_m. \quad (17)$$

In the diffusion approximation, where  $D_m = 0$  for  $m \geq 2$ , one obtains from (17) the relation between the diffusion coefficient  $D_1$  and  $\Delta$  (see [28])

$$D_1 \approx \Delta. \quad (18)$$

This relation was also obtained in Ref. [39].

According to (13) and (17), we have the following small- $\nu$  expansion for the eigenvalue of the BFKL kernel

$$j - 2 = \sum_{m=1}^{\infty} D_m \left( (-\nu^2)^m - 1 \right), \quad (19)$$

where  $\nu^2$  is related to  $\gamma$  according to eq. (15)

$$\nu^2 = -\left(\frac{j}{2} - \gamma\right)^2. \quad (20)$$

On the other hand, due to the ADS/CFT correspondence the string energies  $E$  in dimensionless units are related to the anomalous dimensions  $\gamma$  of the local operators as follows [19, 25]

$$E^2 = (j + \Gamma)^2 - 4, \quad \Gamma = -2\gamma \quad (21)$$

and therefore we can obtain from (20) the relation between the parameter  $\nu$  for the principal series of unitary representations of the Möbius group and the string energy  $E$

$$\nu^2 = -\left(\frac{E^2}{4} + 1\right). \quad (22)$$

This expression for  $\nu^2$  can be inserted in the r.h.s. of Eq. (19) leading to the following expression for the Regge trajectory of the graviton in the anti-de-Sitter space

$$j - 2 = \sum_{m=1}^{\infty} D_m \left[ \left(\frac{E^2}{4} + 1\right)^m - 1 \right]. \quad (23)$$

Note [28], that due to (22) expression (7) for the eigenvalue of the BFKL kernel in the diffusion approximation (18)

$$j = j_0 - \Delta \nu^2 = 2 - \Delta(\nu^2 + 1), \quad (24)$$

is equivalent to the linear graviton Regge trajectory

$$j = 2 + \frac{\alpha'}{2}t, \quad \alpha't = \Delta E^2/2, \quad (25)$$

where its slope  $\alpha'$  and the Mandelstam invariant  $t$ , defined in the 10-dimensional space, equal

$$\alpha' = \frac{R^2}{2}, \quad t = E^2/R^2 \quad (26)$$

and  $R$  is the radius of the anti-de-Sitter space.

Now we return to the eq. (23) in general case. We assume below, that it is valid also at large  $j$  and large  $\lambda$  in the region

$$1 \ll j \ll \sqrt{\lambda}, \quad (27)$$

where the strong coupling calculations of energies were performed [32, 35]. Comparing the l.h.s. and r.h.s. of (23) at large  $j$  values gives us the coefficients  $D_m$  and  $\Delta$  (see Appendix A).<sup>3</sup>

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<sup>3</sup> When this paper was almost prepared for publication, we found the article [40] containing some of our results (see discussions in Appendix A).

## 4 Graviton Regge trajectory and Pomeron intercept

The coefficients  $D_1$  and  $D_2$  at large  $\lambda$  can be written as follows <sup>4</sup>

$$D_1 = \frac{2}{\sqrt{\lambda}} \left( 1 - \frac{2a_{01}}{\sqrt{\lambda}} \right), \quad D_2 = -\frac{8a_{10}}{\lambda^{3/2}}, \quad (28)$$

where  $a_{01}$  and  $a_{10}$  are calculated in Appendix A

$$a_{01} = -\frac{1}{4}, \quad a_{10} = \frac{3}{8}. \quad (29)$$

As a result, we find eigenvalue (23) of the BFKL kernel at large  $\lambda$  in the form of the nonlinear Regge trajectory of the graviton in the anti-de-Sitter space

$$j - 2 = D_1 \frac{E^2}{4} + D_2 \left[ \left( \frac{E^2}{4} \right)^2 + \frac{E^2}{2} \right], \quad E^2 = 2\alpha' t. \quad (30)$$

Note, that the perturbation theory for the BFKL equation gives this trajectory at small  $\omega = j - 1$  (see eq. (1)), where

$$\gamma_{BFKL} = \frac{1}{2} + i\nu, \quad \nu^2 = -\left( \frac{E^2}{4} + 1 \right). \quad (31)$$

However the energy-momentum constraint (14), leading to  $\omega = 1$  at  $E = 0$ , is not fulfilled in the perturbation theory, because at  $\gamma \rightarrow 0$  the right-hand side of (1) contains the pole singularities which should be cancelled after an appropriate resummation of all orders.

Neglecting the term  $D_2 E^2/2 \sim E^2/\lambda^{3/2}$  at  $\lambda \rightarrow \infty$  in comparison with a larger correction  $a_{01} E^2/\lambda$ , we obtain the graviton trajectory (30) in the form

$$j - 2 = \frac{2}{\sqrt{\lambda}} \left( 1 - \frac{2a_{01}}{\sqrt{\lambda}} \right) \frac{E^2}{4} - \frac{8a_{10}}{\lambda^{3/2}} \left( \frac{E^2}{4} \right)^2. \quad (32)$$

Solving this quadratic equation, one can derive with the same accuracy (see [32, 35])

$$\frac{2}{\sqrt{\lambda}} \frac{E^2}{4} = (j - 2) \left( 1 + 2 \frac{a_{01} + a_{10}(j - 2)}{\sqrt{\lambda}} \right). \quad (33)$$

On the other hand, due to (21) this relation can be written as follows

$$\frac{1}{2\sqrt{\lambda}} (j - 2\gamma)^2 = \frac{2}{\sqrt{\lambda}} + (j - 2) \left( 1 + 2 \frac{a_{01} + a_{10}(j - 2)}{\sqrt{\lambda}} \right) \quad (34)$$

and for  $j - 2 \gg 1/\sqrt{\lambda}$  we have

$$j - 2\gamma = \sqrt{2(j - 2)} \lambda^{1/4} \left[ 1 + \left( \frac{1}{j - 2} + a_{01} + a_{10}(j - 2) \right) \frac{1}{\sqrt{\lambda}} \right]. \quad (35)$$

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<sup>4</sup> Here we consider only the calculation of the  $\lambda^{-1}$  correction to Pomeron intercept. More general results are presented in Appendix A.

In particular, for  $j = 4$  one obtains the anomalous dimension for the Konishi operator  $\gamma = \gamma_K$  [32] (see also Appendix B)

$$2 - \gamma_K = \lambda^{1/4} \left[ 1 + \left( \frac{1}{2} + a_{01} + 2a_{10} \right) \frac{1}{\sqrt{\lambda}} \right] = \lambda^{1/4} \left[ 1 + \frac{1}{\sqrt{\lambda}} \right] \quad (36)$$

in an agreement with eq. (28).

For the anomalous dimension at  $j-2 \sim 1/\sqrt{\lambda}$  from (34) we obtain the square root singularity similar to that appearing at small  $j-1 = \omega_0$  (8)

$$\gamma = -\frac{\lambda^{1/4}}{\sqrt{2}} \left( 1 + \frac{a_{01}}{\sqrt{\lambda}} \right) \left( \sqrt{D_1 + j - 2} - \sqrt{D_1} \right), \quad (37)$$

where  $D_1$  (28) is equal to the correction  $\Delta$  to the graviton trajectory intercept with our accuracy

$$\Delta = D_1 \approx \frac{2}{\sqrt{\lambda}} \left( 1 + \frac{1}{2\sqrt{\lambda}} \right).$$

Note, that in the region  $j-2 < -\Delta$ , the anomalous dimension is complex similar to it in the perturbative regime at  $j-1 < \omega_0$  (8). Moreover, the position of the BFKL singularity of  $\gamma$  at large coupling constants can be found from the calculation of the radius of the divergency of the perturbation theory in  $1/\sqrt{\lambda}$  at small  $j-2$ .

## 5 Anomalous dimension near $j = 2$

At  $j = 2$ , the universal anomalous dimension is zero (14), but its derivative  $\gamma'(2)$  (the slope of  $\gamma$ ) has a nonzero value in the perturbative theory

$$\gamma'(2) = -\frac{\lambda}{24} + \frac{1}{2} \left( \frac{\lambda}{24} \right)^2 - \frac{2}{5} \left( \frac{\lambda}{24} \right)^3 + \frac{7}{20} \left( \frac{\lambda}{24} \right)^4 - \frac{11}{35} \left( \frac{\lambda}{24} \right)^5 + O(\lambda^6), \quad (38)$$

as it follows from exact three-loop calculations [12, 28]. Two last terms were calculated by V. Velizhanin [42] from the explicit results for  $\gamma$  in five loops [15].

To find the slope  $\gamma'(2)$  at large values of the coupling constant we calculate the derivatives of the l.h.s. and r.h.s. of eq. (19) written in the form

$$j - 2 = \sum_{m=1} D_m \left[ \left( \frac{j}{2} - \gamma \right)^{2m} - 1 \right] \quad (39)$$

in the variable  $j$  for  $j = 2$  using  $\gamma(2) = 0$ :

$$1 = \left( 1 - 2\gamma'(2) \right) \sum_{m=1} m D_m \equiv \left( 1 - 2\gamma'(2) \right) \overline{D}_1, \quad (40)$$

where  $\overline{D}_1$  is found in Appendix A in the expansion up to  $m = 3$  (see (A.8)). So we obtain explicitly

$$1 - 2\gamma'(2) = \frac{\sqrt{\lambda}}{2} h_0(\lambda). \quad (41)$$

Here

$$h_0(\lambda) = 1 + \sum_{i=1}^{\infty} \frac{a_{0i}}{(\lambda)^{i/2}} \quad (42)$$

is the leading contribution to the string energy in the limit  $j \ll \sqrt{\lambda}$ , as it is shown in (A.1) and (A.2)).

It is important, that now there is an explicit expression for  $h_0(\lambda)$  (see (A.3)). It was obtained from the Basso result [33] by taking the value of the angular momentum  $J_{an}$  equal to two (see [34]). Substituting (A.3) in (41), we have the closed form for the slope  $\gamma'(2)$

$$\gamma'(2) = -\frac{\sqrt{\lambda}}{4} \frac{I_3(\sqrt{\lambda})}{I_2(\sqrt{\lambda})}, \quad (43)$$

which is in full agreement with predictions (38) of perturbation theory.

Note, that in [12] with the use of two first terms of perturbation theory we suggested the simple quadratic equation to resum  $\gamma'(2)$  in all orders

$$\frac{\lambda}{24} = -\gamma'(2) + \frac{1}{2}(\gamma'(2))^2. \quad (44)$$

It turns out, that the solution of this quadratic equation indeed interpolates  $\gamma'(2)$  between weak and strong coupling regimes rather well (see ref. [28]).

In particular, for large  $\lambda$  values, the equation (44) leads to

$$\gamma'(2) = -\frac{\sqrt{\lambda}}{2\sqrt{3}} + 1 + O\left(\frac{1}{\sqrt{\lambda}}\right), \quad (45)$$

whereas the strong coupling expansion of the explicit result (43) is given below

$$\gamma'(2) = -\frac{\sqrt{\lambda}}{4} \left( 1 - \frac{5}{2} \frac{1}{\sqrt{\lambda}} + \frac{15}{8} \frac{1}{\lambda} + \frac{15}{8} \frac{1}{\lambda^{3/2}} + O\left(\frac{1}{\lambda^2}\right) \right). \quad (46)$$

## 6 Numerical analysis of the Pomeron intercept $j_0(\lambda)$

Let us obtain an unified expression for the position of the Pomeron singularity  $j_0 = 1 + \omega_0$  for arbitrary values of  $\lambda$ , using an interpolation between weak and strong coupling regimes.

It is convenient to replace  $\omega_0$  with the new variable  $t$  as follows

$$t_0 = \frac{2\omega_0}{1 - \omega_0^2}, \quad \omega_0 = \frac{\sqrt{1 + t_0^2} - 1}{t_0}. \quad (47)$$

This variable has the asymptotic behavior  $t_0 \sim \lambda$  at  $\lambda \rightarrow 0$  and  $t_0 \sim \sqrt{\lambda}/2$  at  $\lambda \rightarrow \infty$  similar to the case of the cusp anomalous dimension (see, for example, [11]). So, following the method of Refs. [11, 12, 41], we shall write a simple algebraic equation for  $t_0 = t_0(\lambda)$  whose solution will interpolate  $\omega_0$  for the full  $\lambda$  range.

We choose the equation of the form

$$k_0(\lambda) = k_1(\lambda)t_0 + k_2(\lambda)t_0^2, \quad (48)$$



where the following anzatz for the coefficinets  $k_0$ ,  $k_1$  and  $k_2$  is used:

$$k_0(\lambda) = \beta_0\lambda + \alpha_0\lambda^2, \quad k_1(\lambda) = \beta_1 + \alpha_1\lambda, \quad k_2(\lambda) = \gamma_2\lambda^{-1} + \beta_2 + \beta_2\lambda. \quad (49)$$

Here  $\gamma_2$ ,  $\alpha_i$  and  $\beta_i$  ( $i = 0, 1, 2$ ) are free parameters, which are fixed using the known asymptotics of  $\omega_0$  at  $\lambda \rightarrow 0$  and  $\lambda \rightarrow \infty$ .

The solution of quadratic equation (48) is given below

$$t_0 = \frac{\sqrt{k_1^2 + 4k_0k_2} - k_1}{2k_2}. \quad (50)$$

To fix the parameters  $\gamma_2$ ,  $\alpha_i$  and  $\beta_i$  ( $i = 0, 1, 2$ ), we use two known coefficients for the weak coupling expansion of  $\omega_0$ :

$$\omega_0 = \tilde{e}_1\lambda + \tilde{e}_2\lambda^2 + \tilde{e}_3\lambda^3 + \dots \quad (\text{at } \lambda \rightarrow 0) \quad (51)$$

with

$$\tilde{e}_1 = \frac{4 \ln 2}{\pi^2} \approx 0.28092, \quad \tilde{e}_2 = -\tilde{d}_1 \frac{7.5912}{16\pi^2} \approx -0.01350 \quad (52)$$

and first four terms of its strong coupling expansion

$$\omega_0 = 1 - \Delta, \quad \Delta = \frac{2}{\sqrt{\lambda}} \left( 1 + \frac{\tilde{t}_1}{\sqrt{\lambda}} + \frac{\tilde{t}_2}{\lambda} + \frac{\tilde{t}_3}{\lambda^{3/2}} + \frac{\tilde{t}_4}{\lambda^2} + \dots \right) \quad (\text{at } \lambda \rightarrow \infty) \quad (53)$$

with (see below Eq. (62))

$$\tilde{t}_1 = \frac{1}{2}, \quad \tilde{t}_2 = -\frac{1}{8}, \quad \tilde{t}_3 = -1 - 3\zeta_3, \quad \tilde{t}_4 = 2a_{12} - \frac{145}{128} - \frac{9}{2}\zeta_3. \quad (54)$$

The coefficients  $\tilde{e}_3$  and  $\tilde{t}_4$  are unknown but we estimate them later from the interpolation.

Then, for the weak and strong coupling expansions of  $t$  one obtains

$$t_0 = e_1\lambda + e_2\lambda^2 + e_3\lambda^3 + \dots, \quad (\text{when } \lambda \rightarrow 0), \quad (55)$$

$$t_0 = \frac{\sqrt{\lambda}}{2} \left( 1 - \frac{t_1}{\sqrt{\lambda}} - \frac{t_2}{\lambda} - \frac{t_3}{\lambda^{3/2}} - \frac{t_4}{\lambda^2} \right) + \dots, \quad (\text{when } \lambda \rightarrow \infty), \quad (56)$$

where

$$\begin{aligned} e_1 &= \tilde{e}_1, \quad e_2 = \tilde{e}_2, \quad e_3 = \tilde{e}_3 + \tilde{e}_1^3, \quad t_1 = \tilde{t}_1 + 1 = \frac{3}{2}, \quad t_2 = \tilde{t}_2 - \tilde{t}_1^2 + 1 = -\frac{11}{8}, \\ t_3 &= (\tilde{t}_3 - 2\tilde{t}_2\tilde{t}_1 + \tilde{t}_1^3) - \tilde{t}_1 + 1 = -\frac{1}{4}(1 + 12\zeta_3), \\ t_4 &= (\tilde{t}_4 - 2\tilde{t}_3\tilde{t}_1 - \tilde{t}_2^2 + 3\tilde{t}_2\tilde{t}_1^2 - \tilde{t}_1^4) - (\tilde{t}_2 - 2\tilde{t}_1 + 1) = 2a_{12} + \frac{9}{128} - \frac{3}{2}\zeta_3. \end{aligned} \quad (57)$$

Comparing the l.h.s. and the r.h.s. of Eq. (48) at  $\lambda \rightarrow 0$  and  $\lambda \rightarrow \infty$ , respectively, we derive the following relations

$$\alpha_2 = 4\alpha_0, \quad \alpha_2 = 6\alpha_0, \quad \beta_1 = C_1\alpha_0, \quad \beta_2 = C_2\alpha_0, \quad \gamma_2 = C_3\alpha_0, \quad \beta_0 = [C_1e_1 + C_3e_1^2]\alpha_0 \quad (58)$$

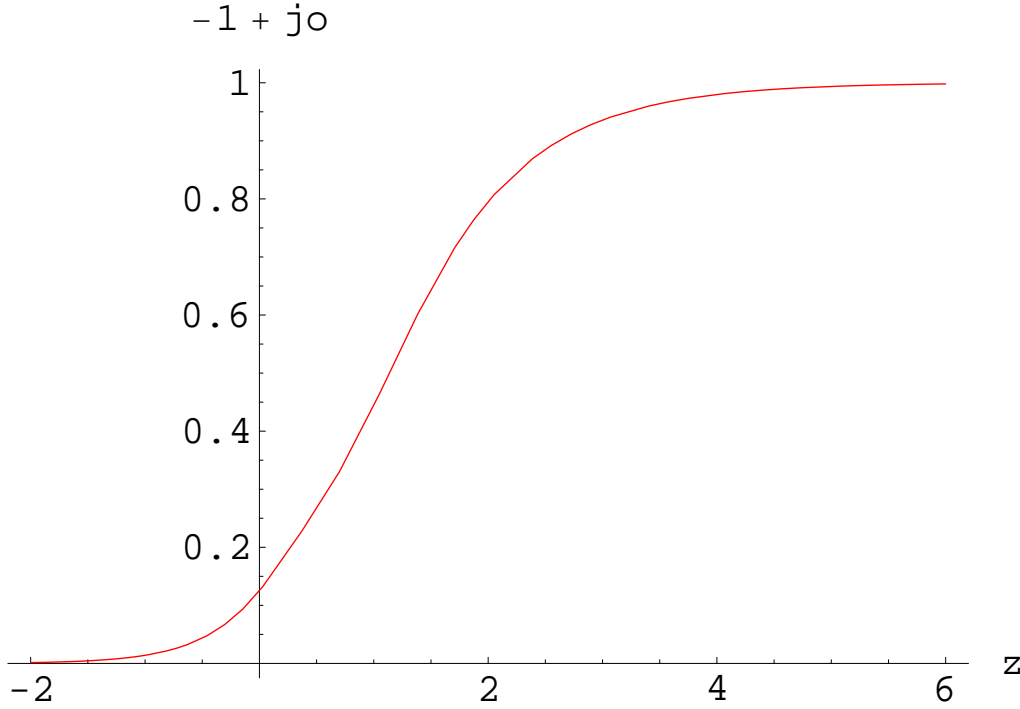


Figure 1: (color-online). The results for  $j_0$  as a function of  $z$  ( $\lambda = 10^z$ ).

with the free parameter  $\alpha_0$  which disappears in the relationship  $k_1/k_2$  and  $k_0/k_2$  and, thus, in the results (50) for  $t_0$ .

Here

$$C_1 \approx -23.99, \quad C_2 \approx -5.71, \quad C_3 \approx 73.65, \quad (59)$$

which lead to the following predictions for the coefficients  $e_3$  and  $t_4$  in (55) and (56)

$$e_3 = -\frac{6e_2 + 2C_2e_1e_2 + 4e_1^2 + C_3e_2^2}{C_1 + 2C_3e_1} \approx -0.01676, \quad t_4 = \frac{121 + 16(C_3 - 3C_1 + 5C_2)}{128} \approx 15.5785 \quad (60)$$

and, respectively, for the corresponding terms in (51), (53) and (54)

$$\tilde{e}_3 \approx -0.03893, \quad \tilde{t}_4 \approx -11.0192, \quad a_{12} \approx 8.1793. \quad (61)$$

Note that the results for the coefficients  $e_3$ ,  $t_4$ ,  $\tilde{e}_3$ ,  $\tilde{t}_4$  and  $a_{12}$  do not depend on the free parameter  $\alpha_0$ .

On Fig. 1, we plot the pomeron intercept  $j_0$  as a function of the coupling constant  $\lambda$ . The behavior of the pomeron intercept  $j_0$  shown in Fig.1 is similar to that found in QCD with some additional assumptions (see ref. [30]).

## 7 Conclusion

We found the intercept of the BFKL pomeron at weak and strong coupling regimes in the  $\mathcal{N} = 4$  Super-symmetric Yang-Mills model.

At large couplings  $\lambda \rightarrow \infty$ , the correction  $\Delta$  for the Pomeron intercept  $j_0 = 2 - \Delta$  has the form (see Appendix (A.21))

$$\Delta = \frac{2}{\lambda^{1/2}} \left[ 1 + \frac{1}{2\lambda^{1/2}} - \frac{1}{8\lambda} - (1 + 3\zeta_3) \frac{1}{\lambda^{3/2}} + \left( 2a_{12} - \frac{145}{128} - \frac{9}{2}\zeta_3 \right) \frac{1}{\lambda^2} + O\left(\frac{1}{\lambda^{5/2}}\right) \right]. \quad (62)$$

The anomalous dimension has a square-root singularity at the value of the BFKL intercept both in the weak and strong coupling regimes. This value is related to the radius of convergency of perturbation theory in  $\lambda$  and  $1/\sqrt{\lambda}$  near the points  $j_0 = 1$  and  $j_0 = 2$ , respectively.

The fourth corrections in (62) contain unknown coefficient  $a_{12}$ , which will be obtained after the evaluation of spinning folded string on the two-loop level. Some estimations were given in Section 6.

The slope of the universal anomalous dimension at  $j = 2$  known by the direct calculations [42] up to the fifth order of perturbation theory can be written as follows

$$\gamma'(2) = - \frac{\sqrt{\lambda}}{4} \frac{I_3(\sqrt{\lambda})}{I_2(\sqrt{\lambda})}, \quad (63)$$

according to the well known Basso result [33] for local operators of an arbitrary twist.

## Acknowledgments.

This work was supported in part by the Alexander von Humboldt fellowship and by RFBR grant (A.V.K.). We are grateful to Arkady Tseytlin and Dmitro Volin for useful discussions.

## A Appendix

Here we discuss coefficients  $D_m$  and the Pomeron intercept  $2 - \Delta$  using expression (23) at comparatively large  $j$  in the region  $j \ll \sqrt{\lambda}$ .

### A.1 String energy at $1 \ll j \ll \sqrt{\lambda}$

The recent results for the string energies [34] in the region restricted by inequalities (27) can be presented in the form <sup>5</sup>

$$\frac{E^2}{4} = \sqrt{\lambda} \frac{S}{2} \left[ h_0(\lambda) + h_1(\lambda) \frac{S}{\sqrt{\lambda}} + h_2(\lambda) \frac{S^2}{\lambda} \right] + O(S^{7/2}), \quad (A.1)$$

where

$$h_i(\lambda) = a_{i0} + \frac{a_{i1}}{\sqrt{\lambda}} + \frac{a_{i2}}{\lambda} + \frac{a_{i3}}{\sqrt{\lambda^3}} + \frac{a_{i2}}{\lambda^2}. \quad (A.2)$$

The contribution  $\sim \sqrt{S}$  can be extracted directly from the Basso result [33] taking  $J_{an} = 2$  according to [34]:

$$h_0(\lambda) = \frac{I_3(\sqrt{\lambda})}{I_2(\sqrt{\lambda})} + \frac{2}{\sqrt{\lambda}} = \frac{I_1(\sqrt{\lambda})}{I_2(\sqrt{\lambda})} - \frac{2}{\sqrt{\lambda}}, \quad (A.3)$$

---

<sup>5</sup>Here we put  $S = j - 2$ , which in particular is related to the use of the angular momentum  $J_{an} = 2$  in calculations of Refs [32, 35].

where  $I_k(\sqrt{\lambda})$  is the modified Bessel functions. It leads to the following values of coefficients  $a_{0i}$

$$a_{00} = 1, \quad a_{01} = -\frac{1}{2}, \quad a_{02} = a_{03} = \frac{15}{8}, \quad a_{04} = \frac{135}{128} \quad (\text{A.4})$$

The coefficients  $a_{10}$  and  $a_{20}$  come from considerations of the classical part of the folded spinning string corresponding to the twist-two operators <sup>6</sup> (see, for example, [35])

$$a_{10} = \frac{3}{4}, \quad a_{20} = -\frac{3}{16}. \quad (\text{A.5})$$

The one-loop coefficient  $a_{11}$  is found recently in the paper [34] (see also [43]), considering different asymptotical regimes with taking into account the Basso result [33]

$$a_{11} = \frac{3}{16}(1 - \zeta_3), \quad (\text{A.6})$$

where  $\zeta_3$  is the Euler  $\zeta$ -function.

All calculations were performed for nonzero values of the angular momentum  $J_{an}$  (really,  $J_{an} = 2$  was used) and are applicable also to the finite  $S$  values. <sup>7</sup> Moreover, all these coefficients are in a full agreement with numerical  $Y$ -system predictions (see [45, 32] and references therein).

## A.2 Equations for coefficients $D_m$ and the Pomeron intercept $2 - \Delta$

Thus, from expression (A.1) we obtain the following expansions of even powers of  $E$  in the small parameter  $j/\sqrt{\lambda}$

$$\left(\frac{E^2}{4}\right)^2 = \lambda \frac{S^2}{4} \left[ h_0^2(\lambda) + 2h_0h_1(\lambda) \frac{S}{\sqrt{\lambda}} \right], \quad \left(\frac{E^2}{4}\right)^3 = \lambda^{3/2} \frac{S^3}{8} h_0^3(\lambda). \quad (\text{A.7})$$

Comparing the coefficients in the front of  $S$ ,  $S^2$  and  $S^3$  in the l.h.s. and r.h.s of (23), we derive the equations

$$1 = \frac{\sqrt{\lambda}}{2} h_0 \overline{D}_1, \quad \overline{D}_1 = (D_1 + 2D_2 + 3D_3), \quad (\text{A.8})$$

$$0 = \frac{1}{2} h_1 \overline{D}_1 + \frac{\lambda}{4} h_0^2 \overline{D}_2, \quad \overline{D}_2 = (D_2 + 3D_3), \quad (\text{A.9})$$

$$0 = \frac{1}{2\sqrt{\lambda}} h_2 \overline{D}_1 + \frac{\sqrt{\lambda}}{4} h_0 h_1 \overline{D}_2 + \frac{\lambda^{3/2}}{8} h_0^3 D_3. \quad (\text{A.10})$$

Their perturbative solution leads is given below

$$\overline{D}_1 = \frac{2}{\sqrt{\lambda}} \frac{1}{h_0} \quad \overline{D}_2 = -\frac{2}{\lambda} \frac{h_1}{h_0^2} \overline{D}_1 = -\frac{4}{\lambda^{3/2}} \frac{h_1}{h_0^3}, \quad (\text{A.11})$$

$$D_3 = \frac{4}{\lambda^2} \frac{2h_1^2 - h_0h_2}{h_0^4} \overline{D}_1 = \frac{8}{\lambda^{5/2}} \frac{2h_1^2 - h_2h_0}{h_0^5}. \quad (\text{A.12})$$

---

<sup>6</sup> We are grateful to Arkady Tseitin for explaining this point.

<sup>7</sup> The previous calculations [44] were done with the zero values of the angular momentum  $J_{an}$  and cannot be directly applied for the finite  $S$  values. We are grateful to Arkady Tseitin for explaining this point to us.

and, correspondingly,

$$D_2 = \overline{D}_2 - 3D_3, \quad D_1 = \overline{D}_1 - 2\overline{D}_2 + 3D_3. \quad (\text{A.13})$$

Finally, we obtain the correction  $\Delta$  to the Pomeron intercept in the form

$$\begin{aligned} \Delta &= D_1 + D_2 + D_3 = \overline{D}_1 - \overline{D}_2 + D_3 \\ &= \frac{2}{\sqrt{\lambda}} \frac{1}{h_0^2} + \frac{4}{\lambda^{3/2}} \frac{h_1}{h_0^3} + \frac{8}{\lambda^{5/2}} \frac{2h_1^2 - h_2 h_0}{h_0^5}, \end{aligned} \quad (\text{A.14})$$

where the  $\lambda$ -dependence of parameters  $h_i$  is given in Eqs. (A.2) and (A.3).

### A.3 Strong coupling expansions of $D_m$ and $\Delta$

Using expressions (A.4)-(A.6) we have

$$D_3 = \frac{8r_3}{\lambda^{5/2}} + O\left(\frac{1}{\lambda^{7/2}}\right), \quad \overline{D}_2 = -\frac{4}{\lambda^{3/2}} \left[ \overline{c}_2 + \frac{\overline{c}_3}{\lambda^{1/2}} + \frac{\overline{c}_4}{\lambda} + O\left(\frac{1}{\lambda^{3/2}}\right) \right], \quad (\text{A.15})$$

$$\overline{D}_1 = \frac{2}{\lambda^{1/2}} \left[ 1 + \frac{\overline{d}_1}{\lambda^{1/2}} + \frac{\overline{d}_2}{\lambda} + \frac{\overline{d}_3}{\lambda^{3/2}} + \frac{\overline{d}_4}{\lambda^2} + O\left(\frac{1}{\lambda^{5/2}}\right) \right], \quad (\text{A.16})$$

where

$$\begin{aligned} \overline{c}_2 &= a_{10} = \frac{3}{4}, \quad \overline{c}_3 = a_{11} - 3a_{10}a_{01} = \frac{3}{16}(7 - 8\zeta_3), \quad r_3 = 2a_{10}^2 - a_{20} = \frac{21}{16}, \\ \overline{c}_4 &= a_{12} + 3a_{10}(2a_{01}^2 - a_{02}) - 3a_{11}a_{01} = a_{12} - \frac{9}{16}(5 + 4\zeta_3) \end{aligned} \quad (\text{A.17})$$

and

$$\begin{aligned} \overline{d}_1 &= -2a_{01} = \frac{1}{2}, \quad \overline{d}_2 = 2a_{01}^2 - a_{02} = -\frac{13}{8}, \quad \overline{d}_3 = 2a_{01}a_{02} - a_{01}^3 - a_{03} = -\frac{29}{8}, \\ \overline{d}_4 &= a_{01}^4 - 3a_{01}^2a_{02} + 2a_{01}a_{03} + a_{02}^2 - a_{04} = -\frac{97}{128}. \end{aligned} \quad (\text{A.18})$$

Here  $a_{02}, a_{12}, a_{03}$  and  $a_{04}$  are parameters which should be calculated in future at two, three and four loops of the string perturbation theory. It is important, that the coefficients  $D_k$  tend to zero at large  $\lambda$  as  $\lambda^{-n+1/2}$

Analogously, we can obtain expressions for  $D_2, D_1$  and  $\Delta$ :

$$D_2 = -\frac{4}{\lambda^{3/2}} \left[ c_2 + \frac{c_3}{\lambda^{1/2}} + \frac{c_4}{\lambda} + O\left(\frac{1}{\lambda^{3/2}}\right) \right], \quad (\text{A.19})$$

$$D_1 = \frac{2}{\lambda^{1/2}} \left[ 1 + \frac{d_1}{\lambda^{1/2}} + \frac{d_2}{\lambda} + \frac{d_3}{\lambda^{3/2}} + \frac{d_4}{\lambda^2} + O\left(\frac{1}{\lambda^{5/2}}\right) \right], \quad (\text{A.20})$$

$$\Delta = \frac{2}{\lambda^{1/2}} \left[ 1 + \frac{\hat{d}_1}{\lambda^{1/2}} + \frac{\hat{d}_2}{\lambda} + \frac{\hat{d}_3}{\lambda^{3/2}} + \frac{\hat{d}_4}{\lambda^2} + O\left(\frac{1}{\lambda^{5/2}}\right) \right], \quad (\text{A.21})$$

where

$$c_2 = \overline{c}_2, \quad c_3 = \overline{c}_3, \quad c_4 = \overline{c}_4 + 6r_3, \quad d_1 = \overline{d}_1 = \hat{d}_1, \quad (\text{A.22})$$

$$d_2 = \overline{d}_2 + 4\overline{c}_2, \quad d_3 = \overline{d}_3 + 4\overline{c}_3, \quad d_4 = \overline{d}_4 + 4\overline{c}_4 + 12r_3, \quad (\text{A.23})$$

$$\hat{d}_2 = \overline{d}_2 + 2\overline{c}_2, \quad \hat{d}_3 = \overline{d}_3 + 2\overline{c}_3, \quad \hat{d}_4 = \overline{d}_4 + 2\overline{c}_4 + 4r_3 \quad (\text{A.24})$$

and all  $\bar{c}_i$  and  $\bar{d}_i$  are given above in eqs (A.17) and (A.18). So, we have

$$\hat{d}_1 = \frac{1}{2}, \quad \hat{d}_2 = -\frac{1}{8}, \quad \hat{d}_3 = -1 - 3\zeta_3, \quad \hat{d}_4 = 2a_{12} - \frac{145}{128} - \frac{9}{2}\zeta_3. \quad (\text{A.25})$$

Using a similar approach, the coefficients  $\hat{d}_1$  and  $\hat{d}_2$  were found recently in the paper [40]. The corresponding coefficients  $c_{2,0}$  and  $c_{3,0}$  in [40] coincide with our  $\hat{d}_1$  and  $\hat{d}_2$  but in the expression for the Pomeron intercept they contributed with an opposite sign. Further, in the talk of Miguel S. Costa “Conformal Regge Theory” on IFT Workshop “Scattering Amplitudes in the Multi-Regge limit” (Universidad Autonoma de Madrid, 24 - 26 Oct 2012) (see <http://www.ift.uam.es/en/node/3985>) the sign of these contributions to the Pomeron intercept was correct but there is a misprint the definition of the parameter of expansion. Note, however, that we have the next term  $\hat{d}_3$  in the strong coupling expansion.

## B Appendix

We apply Eqs. (19) and (20) with  $j = 4$  (and/or  $S = 2$ ) and  $D_i$  ( $i = 1, 2, 3$ ) obtained in Appendix A, to find the large  $\lambda$  asymptotics of the anomalous dimension of the Konishi operator. So, it obeys to the equation

$$2 = \sum_{m=1} D_m (x^m - 1), \quad x \equiv (2 - \gamma_K)^2 \quad (\text{B1})$$

1. It is convenient to consider firstly the particular case, when  $\bar{D}_2 = \bar{D}_3 = 0$  and, thus,  $D_1 = \bar{D}_1 = 2/\sqrt{\lambda}h_0$ . So, we have

$$2 = \bar{D}_1(x - 1) \quad (\text{B2})$$

and

$$x = \frac{2}{\bar{D}_1} + 1 = \sqrt{\lambda}h_0 + 1, \quad (\text{B3})$$

where  $h_0$  has the closed form (A.3). So, the anomalous dimension  $\gamma_K$  can be represented as

$$2 - \gamma_K = (\sqrt{\lambda}h_0 + 1)^{1/2} \approx \lambda^{1/4} \left( \sqrt{h_0} + \frac{1}{2\sqrt{\lambda}\sqrt{h_0}} - \frac{1}{8\lambda h_0^{3/2}} + O\left(\frac{1}{\lambda^2}\right) \right). \quad (\text{B4})$$

For the case of the classic string, where  $h_0 = 1$ , i.e.  $a_{00} = 1$  and  $a_{0i} = 0$  ( $i \geq 1$ ), we reconstruct well-known results <sup>8</sup>

$$2 - \gamma_K \approx \lambda^{1/4} \left( 1 + \frac{1}{2\sqrt{\lambda}} - \frac{1}{8\lambda} + O\left(\frac{1}{\lambda^{3/2}}\right) \right). \quad (\text{B5})$$

For the exact values of  $h_0$  done in Eqs. (A.2) and (A.4), we have

$$\begin{aligned} 2 - \gamma_K &\approx \lambda^{1/4} \left( 1 + \frac{1 + a_{01}}{2\sqrt{\lambda}} + \frac{1}{2\lambda} \left[ a_{02} - \frac{(1 + a_{01})^2}{4} \right] + O\left(\frac{1}{\lambda^{3/2}}\right) \right) \\ &= \lambda^{1/4} \left( 1 + \frac{1}{4\sqrt{\lambda}} + \frac{29}{32\lambda} + O\left(\frac{1}{\lambda^{3/2}}\right) \right). \end{aligned} \quad (\text{B6})$$

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<sup>8</sup>We should remind that our anomalous dimension  $\gamma_K$  has the additional factor  $-1/2$ , i.e.  $\gamma_K = -\gamma_K^{\text{standard}}/2$ .

**2.** In the case when all  $\overline{D}_i$  ( $i = 1, 2, 3$ ) are nonzero, it is convenient to represent the solution of the equation (B1) in the following form

$$x = \sqrt{\lambda}h_0 + 1 + x_1 + \frac{x_2}{\sqrt{\lambda}}. \quad (\text{B7})$$

Expanding  $\overline{D}_i$  in the inverse series of  $\sqrt{\lambda}$  and compare the coefficients in the front of  $\lambda^0$  and  $1/\sqrt{\lambda}$ , we have

$$x_1 = 2a_{10}, \quad x_2 = 2a_{11} + 4a_{20}. \quad (\text{B8})$$

So, the solution of the equation (B7) with the coefficients (B8) has the form

$$2 - \gamma_K \approx \lambda^{1/4} \left( 1 + \frac{a_{01} + 1 + 2a_{10}}{2\sqrt{\lambda}} + \frac{1}{2\lambda} \left[ a_{02} + 2a_{11} + 4a_{20} - \frac{(1 + a_{01} + 2a_{10})^2}{4} \right] + O\left(\frac{1}{\lambda^{3/2}}\right) \right). \quad (\text{B9})$$

Using Eq.s (A.4)-(A.6) the exact values of  $a_{ij}$ , we have

$$2 - \gamma_K \approx \lambda^{1/4} \left( 1 + \frac{1}{\sqrt{\lambda}} + \frac{1}{4\lambda} [1 - 6\zeta_3] + O\left(\frac{1}{\lambda^{3/2}}\right) \right) \quad (\text{B10})$$

We would like to note that our coefficient in the front of  $\lambda^{-1/4}$  is equal to 1, which in an agreement with calculations performed in [45, 32, 35]. Further, the coefficient in front of  $\lambda^{-3/4}$  agrees with the results of [34] (see also Refs. [43] and [46]).

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